Error Propagation in two-sensor 3D position estimation

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Abstract

The accuracy of 3D position estimation using two angles-only sensors (such as passive optical imagers) is investigated. Beginning with the basic multi-sensor triangulation equations used to estimate a 3D target position, error propagation equations are derived by taking the appropriate partial derivatives with respect to various measurement errors. Next the concept of gaussian measurement error is introduced and used to relate the standard deviation of various measurement errors to the standard deviation of the target 3D position estimate. Plots of the various error propagation coefficients are generated. These analytical results are verified using a Monte-Carlo statistical approach. The result is a set of equations and graphs useful for designing a two-sensor system to meet a required accuracy specification.

Key words: triangulation, 3D position estimation, angles-only, multi-sensor, passive-ranging, error propagation

1. Introduction

Passive, angles-only sensors are used in a number of fields, in different ways, to determine the 3D position of objects. These methods range from trilateration [1-3] which uses time of arrival of a unique signal from an object at multiple fixed sensors, to astronomical parallax measurements [4-7] to determine the distance of nearby astronomical objects from their apparent motion relative to distant objects, as the observer (earth) moves, to triangulation. Triangulation can be performed using multiple sensors at known locations to determine the location of a fixed or moving object, or it can be performed using a moving sensor (observing a stationary target) to create a virtual baseline over time [8-10]. The technique of interest in this paper is triangulation between two sensors.

Triangulation between multiple sensors has been used in a number of fields, including particle tracking velocimetry (PTV) for flow analysis [11-18], surveillance [19], range instrumentation [20], obstacle avoidance [21], and automotive crash test analysis. Surprisingly, despite the use of triangulation for 3D position (and hence velocity) estimation in these fields, there has been little analysis of error propagation and sensitivity published. The only substantial effort at error analysis in the PTV field was by Guezennec and Kiritsis [18] in which they generated synthetic images with known particle properties, in order to perform a statistical test of their image processing and triangulation algorithm. However, that work did not address errors in sensor position and orientation. Angus [19] provides a theoretical analysis of range errors (for a specific co-linear three sensor system) due to uncertainties in sensor azimuth angles but assumes the sensor locations are known exactly and does not address elevation angle errors. Sanders-Reed [20] investigated a three sensor system with arbitrary sensor locations, but like Guezennec and Kiritsis, assumed sensor location and pointing were known exactly, and focused instead on the image processing and triangulation errors using synthetically generated imagery having a known "truth" location for targets. Sridhar and Suorsa [21] compare errors in range estimation for two sensor triangulation with those for a single moving sensor using optical flow and object size change as the sensor approaches the object. For the triangulation they assume the sensor location (baseline) is known accurately, and that measurement errors are present in the object location measurement within the image (corresponding to angular pointing errors).

The present work extends this previous work by developing an analytical relationship between sensor location (x,y,z) and Line Of Sight (LOS) orientation (azimuth and elevation angles) measurement errors and the resulting 3D position estimate of the target. Inherent in this analysis is an understanding of how the sensor to target orientation affects the accuracy of the ultimate target position estimation.

Typically, 2 or more sensors are placed at known locations (x,y,z) and are pointed in the general direction of the object which it is desired to measure. The pointing angles are usually azimuth and elevation, measured in a suitable reference frame such as relative to north or in some convenient laboratory frame. While the analysis in this paper is general (applies to radar, electro-optical, or even acoustic sensors), the most common sensor used for this application is an optical sensor using a rectangular focal plane array. Although a roll angle can be included for focal plane array sensors, usually the sensor focal plane array is aligned with the azimuth and elevation axes so the roll angle is zero and can be ignored.

In general, the minimum number of angles-only sensors which can be used to triangulate the 3D position of an object is two. The exception is a stationary target which can be observed at two different times by moving a single sensor and thus creating a synthetic baseline. While multiple (more than 2) sensors can be used to perform the 3D triangulation, in the current paper, we restrict our error analysis to the somewhat simpler case of only 2 sensors. This restriction is justified as the most commonly used geometry.

For a focal plane array sensor, triangulation is performed by taking an image of the object with each sensor. The object location in each image is computed and the pointing angle to the object is determined. This is done using the Incremental Field Of View (IFOV), which is the angular extent of each pixel, and the number of pixels between the target and the optical axis of the system. Thus if a sensor is aligned with an azimuth angle of 45 degrees and the object is found to be 10 pixels to the right of the optical axis with an IFOV of 0.01 degrees per pixel, then the line of sight azimuth to the target is 45.1 degrees. For the analysis in this paper, any errors in the IFOV, or detected target location on the focal plane, are considered part of the overall pointing angle.

Practical implementations must also deal with several other considerations which are beyond the scope of this paper. One must either synchronize the frame times from multiple sensors, or extrapolate data from the sample times of one sensor, to the sample times of another sensor. For short focal length lenses, the conversion from focal plane pixel coordinates to line of sight angular coordinates is complicated by geometric optical distortion. For the purposes of this paper, these can all be considered basic line of sight measurement error without consideration to the origin of the error. However, for a practical experimenter, an understanding and control of these errors can be critical.

The following sections will provide an overview of the basic triangulation theory, followed by a mathematical derivation of the error propagation equations. Next the assumption of measurement errors having a gaussian statistical distribution is introduced. Error propagation coefficients are computed and plotted over a range of sensor to target orientations. Following the error propagation coefficient derived graphs, a section with Monte-Carlo results is presented and shown to be in agreement with the graphs. Based on the error propagation equations and graphs, experimental setup considerations are discussed.

2. Theory

2.1 3D position estimation

This section presents a brief summary of theory and notation for basic multi-sensor 3D triangulation. We use the coordinate system shown in figure 1, where the target location is shown with the subscript ‘t’ and the sensors are shown with subscript ‘i’. A minimum of 2 sensors is required to produce a solution and if more than two sensors are available, the problem is over-constrained and becomes a least squared problem.

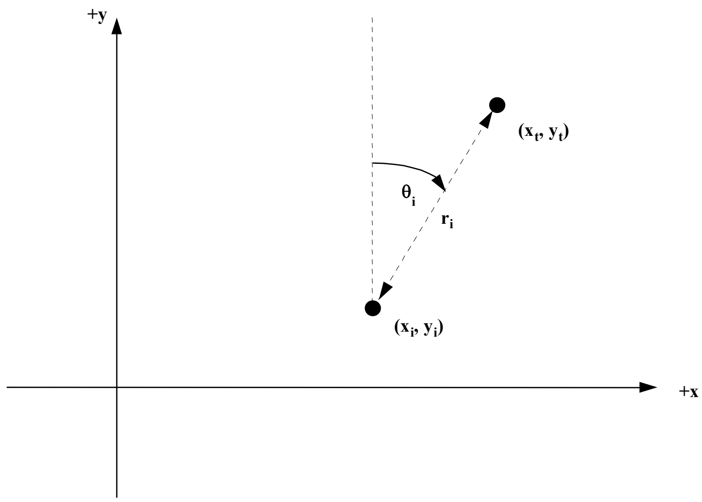

Figure 1. 3D Triangulation Coordinate System. Positive Z axis is out of the paper.

The target location is specified as (x_t,y_t,z_t). The sensors are specified by both their location and their pointing angles. The sensor location is (x_i,y_i,z_i), while the azimuth is θ_i and the elevation is φ_i. Azimuth is measured clockwise from the positive y axis and takes on values from 0 to 2π radians, while elevation is measured from the x-y plane and takes on values between $-\pi$ and $+\pi$. The x-y plane distance from sensor i to the target is r_i.

The azimuth angle is related to the x-y plane coordinates of the target and sensor by:

$$\tan\theta_i = \frac{x_t - x_i}{y_t - y_i} \qquad (1)$$

The target z location is related to the sensor z location, elevation angle, and x-y plane separation distance by:

$$z_t = r_i \tan\varphi_i + z_i \qquad (2)$$

where:

$$r_i = \sqrt{(x_i - x_t)^2 + (y_i - y_t)^2} \qquad (3)$$

The general solution to equation (1), for N sensors, is a least squares solution, best written in matrix notation as:

$$\begin{bmatrix} \cdot \\ x_i - y_i \tan\theta_i \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ 1 & -\tan\theta_i \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \qquad (4)$$

To compute z_t, we just compute the value for each sensor and average the values:

$$\langle z_t \rangle = \langle r_i \tan\varphi_i + z_i \rangle \qquad (5)$$

For a two sensor problem, we can solve equation (1) explicitly using subscripts 1 and 2 to denote the two sensors. In this case we arrive at:

$$x_t = [x_2 \tan\theta_1 - x_1 \tan\theta_2 + (y_1 - y_2)\tan\theta_1 \tan\theta_2](\tan\theta_1 - \tan\theta_2)^{-1} \quad (6)$$

$$y_t = [y_1 \tan\theta_1 - y_2 \tan\theta_2 + x_2 - x_1](\tan\theta_1 - \tan\theta_2)^{-1} \quad (7)$$

Equations (6) and (7) are valid only for the two sensor case. If one uses more than two sensors, equation (4) is the simplest solution for x_t and y_t,.

2.2 Error propagation for 2 sensors

In this section, we derive the error propagation equations for a two sensor system. First we derive the basic error propagation equations giving the error in the final estimates of the target location for specific errors in the sensor position and pointing angles. Following this, we add the assumption of gaussian random measurement errors.

To develop the relationship between measurement error and the estimated value of x_t in equation (6), we perform a Taylor series expansion about the (unknown) true values, keeping only terms to first order, to write the error propagation equation for x_t:

$$\delta x_t \approx \frac{\partial x_t}{\partial x_1}\delta x_1 + \frac{\partial x_t}{\partial x_2}\delta x_2 + \frac{\partial x_t}{\partial y_1}\delta y_1 + \frac{\partial x_t}{\partial y_2}\delta y_2 + \frac{\partial x_t}{\partial \theta_1}\delta\theta_1 + \frac{\partial x_t}{\partial \theta_2}\delta\theta_2 \quad (8)$$

Assuming the errors are small, we have ignored higher order terms. We call this the error propagation equation because, to first order, it describes the dependence of the final answer (e.g. x_t) on errors in each of the measured quantities. For example, to see how x_t varies with errors in the measurement of x_1 (x position of sensor 1), we multiply the x_1 error estimate by the error coefficient, $\partial x_t / \partial x_1$. Looking at the results below, we see that the x_1 error coefficient depends on the separation angle between the two sensors ($\tan\theta_1$-$\tan\theta_2$) and on $\tan\theta_2$.

The various partial derivatives in this equation are:

$$\frac{\partial x_t}{\partial x_1} = -\frac{\tan\theta_2}{\tan\theta_1 - \tan\theta_2}; \qquad \frac{\partial x_t}{\partial x_2} = \frac{\tan\theta_1}{\tan\theta_1 - \tan\theta_2}$$

$$\frac{\partial x_t}{\partial y_1} = \frac{\tan\theta_1 \tan\theta_2}{\tan\theta_1 - \tan\theta_2}; \qquad \frac{\partial x_t}{\partial y_2} = -\frac{\tan\theta_1 \tan\theta_2}{\tan\theta_1 - \tan\theta_2}$$

$$\frac{\partial x_t}{\partial \theta_1} = \left[\frac{x_2 + (y_1 - y_2)\tan\theta_2}{\tan\theta_1 - \tan\theta_2} - \frac{x_2 \tan\theta_1 - x_1 \tan\theta_2 + (y_1 - y_2)\tan\theta_1 \tan\theta_2}{(\tan\theta_1 - \tan\theta_2)^2}\right]\sec^2\theta_1$$

$$\frac{\partial x_t}{\partial \theta_2} = \left[\frac{-x_1 + (y_1 - y_2)\tan\theta_1}{\tan\theta_1 - \tan\theta_2} + \frac{x_2 \tan\theta_1 - x_1 \tan\theta_2 + (y_1 - y_2)\tan\theta_1 \tan\theta_2}{(\tan\theta_1 - \tan\theta_2)^2}\right]\sec^2\theta_2 \quad (9)$$

The last two partial derivatives could be rewritten using either the definition $\sec\theta = 1/\cos\theta$, or the identity, $\sec^2\theta - \tan^2\theta = 1$. More importantly, we note $\partial x_t/\partial x_1 + \partial x_t/\partial x_2 = 1$ and $\partial x_t/\partial y_1 + \partial x_t/\partial y_2 = 0$. Thus while the individual error coefficients themselves have some dependence on the absolute coordinates, the appropriate pairs do not. Put another way, by appropriate choice of coordinates, it is possible to reduce the sensitivity of the triangulation to errors in the x position of sensor 1, but only at the cost of an equal increase in the sensitivity to errors in the x position of sensor 2.

Performing a similar Taylor series expansion using equation (7), and again keeping only terms to first order, we can write the error propagation equation to first order for y_t:

$$\delta y_t \approx \frac{\partial y_t}{\partial y_1}\delta y_1 + \frac{\partial y_t}{\partial y_2}\delta y_2 + \frac{\partial y_t}{\partial x_1}\delta x_1 + \frac{\partial y_t}{\partial x_2}\delta x_2 + \frac{\partial y_t}{\partial \theta_1}\delta\theta_1 + \frac{\partial y_t}{\partial \theta_2}\delta\theta_2 \qquad (10)$$

The various partial derivatives for this equation are:

$$\frac{\partial y_t}{\partial y_1} = \frac{\tan\theta_1}{\tan\theta_1 - \tan\theta_2}; \qquad \frac{\partial y_t}{\partial y_2} = -\frac{\tan\theta_2}{\tan\theta_1 - \tan\theta_2}$$

$$\frac{\partial y_t}{\partial x_1} = \frac{-1}{\tan\theta_1 - \tan\theta_2}; \qquad \frac{\partial y_t}{\partial x_2} = \frac{1}{\tan\theta_1 - \tan\theta_2}$$

$$\frac{\partial y_t}{\partial \theta_1} = \left[\frac{y_1}{\tan\theta_1 - \tan\theta_2} - \frac{y_1\tan\theta_1 - y_2\tan\theta_2 + x_2 - x_1}{(\tan\theta_1 - \tan\theta_2)^2}\right]\sec^2\theta_1$$

$$\frac{\partial y_t}{\partial \theta_2} = \left[\frac{-y_2}{\tan\theta_1 - \tan\theta_2} + \frac{y_1\tan\theta_1 - y_2\tan\theta_2 + x_2 - x_1}{(\tan\theta_1 - \tan\theta_2)^2}\right]\sec^2\theta_2 \qquad (11)$$

Again we note that $\partial y_t/\partial x_1 + \partial y_t/\partial x_2 = 0$ and $\partial y_t/\partial y_1 + \partial y_t/\partial y_2 = 1$.

The first order error propagation equation for the horizontal distance r_i between sensor i and the target t is given by:

$$\delta r_i \approx \frac{\partial r_i}{\partial x_i}\delta x_i + \frac{\partial r_i}{\partial x_t}\delta x_t + \frac{\partial r_i}{\partial y_i}\delta y_i + \frac{\partial r_i}{\partial y_t}\delta y_t \qquad (12)$$

But for ∂x_t and ∂y_t in equation (12), we substitute equations (8) and (10), yielding:

$$\begin{aligned}\delta r_i \approx{}& \frac{\partial r_i}{\partial x_i}\delta x_i + \frac{\partial r_i}{\partial x_t}\left[\frac{\partial x_t}{\partial x_1}\delta x_1 + \frac{\partial x_t}{\partial x_2}\delta x_2 + \frac{\partial x_t}{\partial y_1}\delta y_1 + \frac{\partial x_t}{\partial y_2}\delta y_2 + \frac{\partial x_t}{\partial \theta_1}\delta\theta_1 + \frac{\partial x_t}{\partial \theta_2}\delta\theta_2\right] \\ &+ \frac{\partial r_i}{\partial y_i}\delta y_i + \frac{\partial r_i}{\partial y_t}\left[\frac{\partial y_t}{\partial y_1}\delta y_1 + \frac{\partial y_t}{\partial y_2}\delta y_2 + \frac{\partial y_t}{\partial x_1}\delta x_1 + \frac{\partial y_t}{\partial x_2}\delta x_2 + \frac{\partial y_t}{\partial \theta_1}\delta\theta_1 + \frac{\partial y_t}{\partial \theta_2}\delta\theta_2\right]\end{aligned} \qquad (13)$$

The remaining terms which must be evaluated are:

$$\frac{\partial r_i}{\partial x_i} = \frac{x_i - x_t}{\sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}}; \qquad \frac{\partial r_i}{\partial x_t} = -\frac{x_i - x_t}{\sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}}$$

$$\frac{\partial r_i}{\partial y_i} = \frac{y_i - y_t}{\sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}}; \qquad \frac{\partial r_i}{\partial y_t} = -\frac{y_i - y_t}{\sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}} \qquad (14)$$

Using the terms in equations (9), (11), and (14), we can evaluate the error propagation equation (12) for the x-y plane distance between sensor i and the target t.

We are now ready to use equation (2) to write the first order error propagation for z_t:

$$\delta z_t \approx \frac{\partial z_t}{\partial r_i}\delta r_i + \frac{\partial z_t}{\partial \varphi_i}\delta\varphi_i + \frac{\partial z_t}{\partial z_i}\delta z_i \qquad (15)$$

We have already computed δr_i. The remaining partial derivatives are:

$$\frac{\partial z_t}{\partial r_i} = \tan\varphi_i \,; \quad \frac{\partial z_t}{\partial \varphi_i} = r_i \sec^2 \varphi_i \,; \qquad \frac{\partial z_t}{\partial z_i} = 1 \qquad (16)$$

At this point we have evaluated all the error propagation equations for x_t (equation 8), y_t (equation 10), r_i (equation 13) and z_t (equation 15). The various partial derivatives required to evaluate these equations are contained in equations (9), (11), (14) and (16). These can be easily programmed into a computer and evaluated for various sensor to target geometries.

We can make several observations about the propagation of measurement errors from the preceding equations.

1. Estimates of x_t and y_t are independent of any errors in measurement of either the sensor z position (z_i) or elevation angle φ_i (from equations 4, 6, 7).
2. Errors in x_t and y_t arising from measurement errors in sensor position x_i and y_i, depend only on the azimuth angles (specifically, the angle between sensors) and not on the actual sensor (x,y) locations (equations 9, 11).
3. All of the partial derivatives for δx_t and δy_t (equations 9 and 11), contain a ($\tan\theta_1$-$\tan\theta_2$) term in the denominator. This term goes to zero, and hence the error propagation coefficients become infinite when the two tangents are equal. This occurs when $\theta_1 = \theta_2 + k\pi$, for any integer k. The two cases of practical interest occur when $\theta_1 = \theta_2$ or when $\theta_1 = \theta_2 + 180$ degrees.
4. Errors in the measurement of the z-component of a sensor position translate directly into equivalent errors in the z-component of the target position estimate (equation 16).

2.3 Error propagation with gaussian statistics

While equation (8) describes the error propagation from each independent variable to the final estimate of x_t, most measurement errors can be described by a random, gaussian error process. In this case, we can derive an estimate of the standard deviation of the estimated target position (x_t) by squaring equation (8), taking the expectation value, and finally taking the square root. Assuming the errors are not correlated, the result is:

$$\sigma_{x_t} \approx \sqrt{\left(\frac{\partial x_t}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial x_t}{\partial x_2}\sigma_{x_2}\right)^2 + \left(\frac{\partial x_t}{\partial y_1}\sigma_{y_1}\right)^2 + \left(\frac{\partial x_t}{\partial y_2}\sigma_{y_2}\right)^2 + \left(\frac{\partial x_t}{\partial \theta_1}\sigma_{\theta_1}\right)^2 + \left(\frac{\partial x_t}{\partial \theta_2}\sigma_{\theta_2}\right)^2} \qquad (17)$$

where σ_a is the standard deviation in measurements for variable a. This equation summarizes the fact that, in general, measurement errors in independent variables are not correlated and hence do not always add with the same sign. Extension of equation (17) to the case in which two of the variables are correlated is straight-forward. If two variables (v_1 and v_2) are correlated, the expectation value of the cross term $\langle\delta v_1 \delta v_2\rangle$ will be non-zero with value $\rho\sigma_{v1}\sigma_{v2}$, where $-1 \leq \rho \leq 1$. This will lead to an additional term within the square root, of the form $2\frac{\partial x_t}{\partial v_1}\frac{\partial x_t}{\partial v_2}\rho\sigma_{v_1}\sigma_{v_2}$.

Similarly, we can use equation (10) to derive an estimate of the standard deviation of y_t:

$$\sigma_{y_t} \approx \sqrt{\left(\frac{\partial y_t}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial y_t}{\partial x_2}\sigma_{x_2}\right)^2 + \left(\frac{\partial y_t}{\partial y_1}\sigma_{y_1}\right)^2 + \left(\frac{\partial y_t}{\partial y_2}\sigma_{y_2}\right)^2 + \left(\frac{\partial y_t}{\partial \theta_1}\sigma_{\theta_1}\right)^2 + \left(\frac{\partial y_t}{\partial \theta_2}\sigma_{\theta_2}\right)^2} \qquad (18)$$

Equation (13) yields:

$$\sigma_{r_i}^2 \approx \left(\frac{\partial r_i}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \left(\frac{\partial r_i}{\partial y_i}\right)^2 \sigma_{y_i}^2$$

$$+\left(\frac{\partial r_i}{\partial x_t}\right)^2 \left[\left(\frac{\partial x_t}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial x_t}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial x_t}{\partial y_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{\partial x_t}{\partial y_2}\right)^2 \sigma_{y_2}^2 + \left(\frac{\partial x_t}{\partial \theta_1}\right)^2 \sigma_{\theta_1}^2 + \left(\frac{\partial x_t}{\partial \theta_2}\right)^2 \sigma_{\theta_2}^2\right]$$

$$+\left(\frac{\partial r_i}{\partial y_t}\right)^2 \left[\left(\frac{\partial y_t}{\partial y_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{\partial y_t}{\partial y_2}\right)^2 \sigma_{y_2}^2 + \left(\frac{\partial y_t}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial y_t}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial y_t}{\partial \theta_1}\right)^2 \sigma_{\theta_1}^2 + \left(\frac{\partial y_t}{\partial \theta_2}\right)^2 \sigma_{\theta_2}^2\right] \quad (19)$$

Finally, equation (15) gives (with the substitution of $\partial z_t/\partial z_i = 1$, from equation 16):

$$\sigma_{z_t} \approx \sqrt{\left(\frac{\partial z_t}{\partial r_i}\right)^2 \sigma_{r_i}^2 + \left(\frac{\partial z_t}{\partial \varphi_i}\right)^2 \sigma_{\varphi_i}^2 + \sigma_{z_i}{}^2} \quad (20)$$

To proceed from here, we make the assumption that the standard deviation of measurement errors for sensors 1 and 2 are independent of the sensor, and hence equal. We further assume that there is no preferred direction and hence measurement errors of the sensor position will have the same standard deviations in both the x and y direction. These two assumption can be written:

$$\sigma_{x_1} = \sigma_{x_2} = \sigma_{y_1} = \sigma_{y_2} \equiv \sigma_{pos}; \qquad \sigma_{\theta_1} = \sigma_{\theta_2} \equiv \sigma_\theta \quad (21)$$

Using these assumptions we can re-write equations (17), (18), and (19) as follows (equation 20 remains unchanged):

$$\sigma_{x_t} \approx \sqrt{\left[\left(\frac{\partial x_t}{\partial x_1}\right)^2 + \left(\frac{\partial x_t}{\partial x_2}\right)^2 + \left(\frac{\partial x_t}{\partial y_1}\right)^2 + \left(\frac{\partial x_t}{\partial y_2}\right)^2\right]\sigma_{pos}^2 + \left[\left(\frac{\partial x_t}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_t}{\partial \theta_2}\right)^2\right]\sigma_\theta^2} \quad (22)$$

$$\sigma_{y_t} \approx \sqrt{\left[\left(\frac{\partial y_t}{\partial x_1}\right)^2 + \left(\frac{\partial y_t}{\partial x_2}\right)^2 + \left(\frac{\partial y_t}{\partial y_1}\right)^2 + \left(\frac{\partial y_t}{\partial y_2}\right)^2\right]\sigma_{pos}^2 + \left[\left(\frac{\partial y_t}{\partial \theta_1}\right)^2 + \left(\frac{\partial y_t}{\partial \theta_2}\right)^2\right]\sigma_\theta^2} \quad (23)$$

$$\sigma_{r_i}^2 \approx \left\{\begin{array}{l}\left(\frac{\partial r_i}{\partial x_i}\right)^2 + \left(\frac{\partial r_i}{\partial y_i}\right)^2 + \left(\frac{\partial r_i}{\partial x_t}\right)^2\left[\left(\frac{\partial x_t}{\partial x_1}\right)^2 + \left(\frac{\partial x_t}{\partial x_2}\right)^2 + \left(\frac{\partial x_t}{\partial y_1}\right)^2 + \left(\frac{\partial x_t}{\partial y_2}\right)^2\right] \\ + \left(\frac{\partial r_i}{\partial y_t}\right)^2\left[\left(\frac{\partial y_t}{\partial y_1}\right)^2 + \left(\frac{\partial y_t}{\partial y_2}\right)^2 + \left(\frac{\partial y_t}{\partial x_1}\right)^2 + \left(\frac{\partial y_t}{\partial x_2}\right)^2\right]\end{array}\right\}\sigma_{pos}^2 \quad (24)$$

$$+\left\{\left(\frac{\partial r_i}{\partial x_t}\right)^2\left[\left(\frac{\partial x_t}{\partial \theta_1}\right)^2 + \left(\frac{\partial x_t}{\partial \theta_2}\right)^2\right] + \left(\frac{\partial r_i}{\partial y_t}\right)^2\left[\left(\frac{\partial y_t}{\partial \theta_1}\right)^2 + \left(\frac{\partial y_t}{\partial \theta_2}\right)^2\right]\right\}\sigma_\theta^2$$

Equation (20) provides an error estimate for z_t from a single sensor. Since we have two sensors, we compute an estimate of z_t from each sensor and then average them (weighted by their standard deviation given in equation 20) to obtain our best estimate of z_t (equation 5). Since both estimates of z_t depend on the position and pointing parameters of both sensors (through r_i), the error estimate (using equation 15) becomes:

$$\left\langle \sigma_{z_t} \right\rangle = \frac{1}{2}\sqrt{\sigma_{z_t}^2(1) + \sigma_{z_t}^2(2) + 2\frac{\partial z_t}{\partial r_1}\frac{\partial z_t}{\partial r_2}Cov(\delta r_1, \delta r_2)} \qquad (25)$$

where $\sigma_z(1)$ is the result obtained from equation (20) for sensor 1, while $\sigma_z(2)$ is the result obtained for sensor 2. The two partial derivatives are given by equation (16) as simply the tangent of the elevation angle of the respective sensors. We evaluate the covariance, $Cov(\delta r_1, \delta r_2)$ as the expectation value of the product, $\langle \delta r_1 \cdot \delta r_2 \rangle$. After much algebra, we arrive at:

$$Cov(\delta r_1, \delta r_2) = \frac{\partial r_1}{\partial x_t}\frac{\partial r_2}{\partial x_t}\sigma_{x_t}^2 + \frac{\partial r_1}{\partial y_t}\frac{\partial r_2}{\partial y_t}\sigma_{y_t}^2 \qquad (26)$$

The partial derivatives are given in equations (14), while expressions for the standard deviations are given by equations (22) and (23).

Equations (20) – (26) are our final error propagation equations relating the standard deviation of our measurement errors to errors in the final target location estimation. For a two sensor system, we can use equations (3), (5) – (7) to compute an estimate of the target location (x_t, y_t, z_t). Then we can use equations (20) – (26) to provide error estimates for those values, based on estimates of the measurement errors.

3. Results

3.1 Methodology

In this section we numerically examine the impact of sensor location and pointing errors on estimates of target location. This is done by examining the behavior of the coefficients of the sensor position errors (σ_{pos}) and sensor azimuth errors (σ_θ) in the equations for target location (x_t, y_t) and sensor to target range, r_i (equations 22-24). We also examine the behavior of the coefficient of the sensor elevation angle (σ_φ) on estimates of the target z_t location (equation 20). We will see that the dominant effects are the sensor azimuth separation angle (θ_1 - θ_2), the range to the target, and the sensor elevation angle.

In order to compute these coefficients, each of the partial derivatives in equations (9), (11), (14), and (16) were coded in a computer program. The accuracy of the computer implementation was checked by comparing the results for several sets of known inputs against hand calculations performed directly from the equations. Another check involved comparing the value of the partial derivatives with the difference between the known and computed target location when a known error was introduced into equations (2), (4), (6) and (7). A further check of the implementation was performed by plotting several of the partial derivative pairs, such as $\partial x_t/\partial x_1 + \partial x_t/\partial x_2$, versus the azimuth angle between sensor 1 and sensor 2 (θ_1 - θ_2) and checking that they gave the appropriate constant value (e.g. 1 or 0) for all separation angles.

Once the veracity of the computer implementation of the partial derivatives was assured, the error coefficients for the measurement errors in equations (20), (22)-(24) were coded. Again, these were checked against hand calculations using the values of the partial derivatives as input.

To generate the plots in the next section, a computer simulation was developed in which a target is placed at coordinates x_t=50; y_t=100, z_t=2 (50, 100, 2). Next, a sensor 1 azimuth angle θ_1, range r_1, and z axis position z_1 are selected. In most cases, a range of 5 units was used. Using this information, the (x_1, y_1) coordinates of sensor 1, as well as the elevation angle φ_1 are determined. Next, sensor 2 is placed at the same range (r_1=r_2), but is moved in one degree increments around the target, such that the azimuth separation θ_2-θ_1 varies between 1° and 179°. At each increment, the error propagation coefficients are computed. This was typically done for a variety of different sensor 1 azimuth angles. The coefficients of σ_{pos} measurement errors are units of target position error per unit of sensor location error. The coefficients of σ_θ measurement errors are units of target position error per degree of sensor azimuth error.

In order to determine the (x,y) coordinates of each sensor, given the range and the azimuth angle to the target, equations (1) and (3) were solved for $\Delta x = |x_t - x_i|$ and $\Delta y = |y_t - y_i|$. The results are:

$$\Delta y_i = \sqrt{\frac{r_i^2}{\tan^2 \theta_i + 1}} \qquad (27)$$

$$\Delta x_i = \sqrt{r_i^2 - \Delta y_i^2} \qquad (28)$$

$$\tan \varphi_i = \frac{\Delta z_i}{r_i} \qquad (29)$$

Following the explicitly computed error propagation graphs, is a section containing Monte-Carlo results for several different sensor pair configurations. Four sensors are used to generate 6 pairs of sensors. All four sensors are located at a range of 5 units from the target. Sensor 1 is located at an azimuth angle of 40°, sensor 2 at 50°, sensor 3 at 130°, and sensor 4 at 140°. This allows 2 sensor pairs at 10° separation, 1 pair at 80° separation, 2 pairs at 90° separation, and 1 pair at 100° separation.

Equations (6), (7) and (3) are used to compute the target x-y location and the sensor 1 to target range, for each pair of sensors. One thousand sample runs were used, having gaussian distributed measurement errors. The standard deviation of the target x-y position estimate and of the sensor 1 to target range estimate was computed for each sensor pair. The results were compared with the analytical predictions in the graphs (section 3.2).

3.2 Error Propagation Graphs

We begin by examining the behavior of the coefficients of σ_{pos} and σ_θ in equations (22) and (23). Figures 2-5 show plots of the coefficients as a function of the azimuth angle between the sensors (θ_1-θ_2). As can be seen from equations (9) and (11), these coefficients depend not only on the azimuth angle between sensors, but also on the absolute azimuth angle. We examine this dependence by selecting various fixed sensor 1 azimuth angles (θ_1) and then varying the sensor 2 azimuth angle (θ_2).

Examining figures 2 and 3, showing the sensitivity of x and y target position estimates to sensor location measurement errors as function of azimuth separation angle, we see that, as one might expect, they are equivalent with a 90° shift in θ_1 values between the two graphs. In both cases, minimum sensitivity is obtained at an azimuth separation angle of 90°. Target position sensitivity to sensor position measurement error varies from 1:1 at a 90° separation angle to 4:1 at 20° and 160° separation angles.

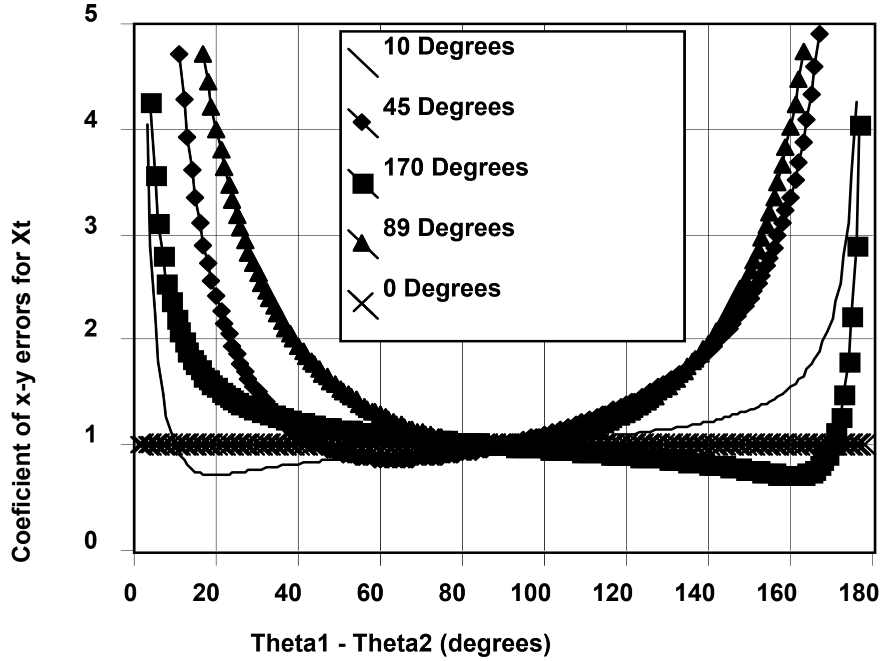

Figure 2. Coefficient of σ_{pos} in equation (22) for σ_x as a function of azimuth separation angle (θ_1 - θ_2) for various values of sensor 1 azimuth angle (θ_1).

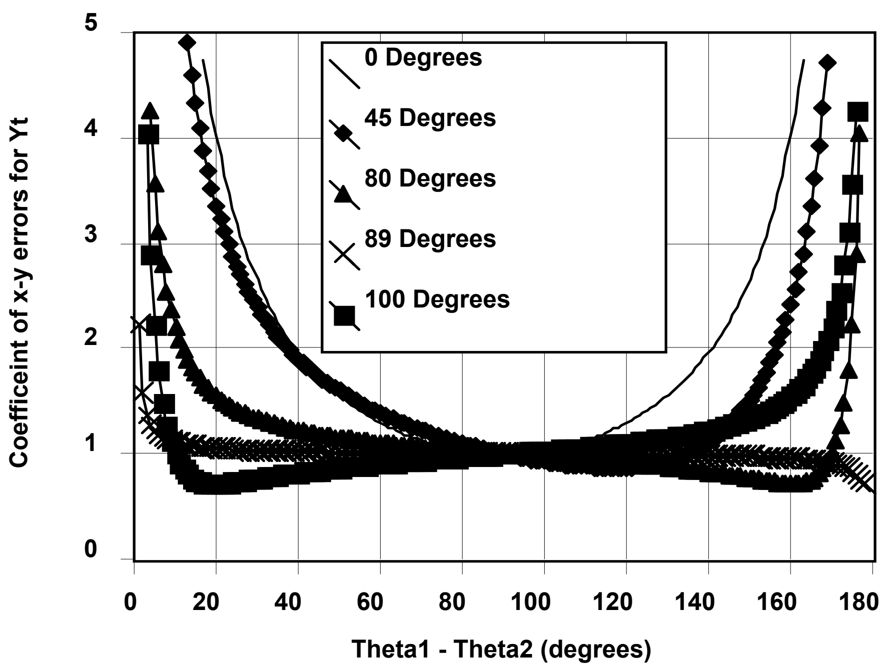

Figure 3. Coefficient of σ_{pos} in equation (23) for σ_y as a function of azimuth separation angle (θ_1 - θ_2) for various values of sensor 1 azimuth angle (θ_1).

Examining figures 4 and 5, showing the sensitivity of x and y target position estimates to sensor azimuth measurement errors (in degrees) as function of azimuth separation angle, we see that, as one might expect, they are equivalent with a 90° shift in θ_1 values between the two graphs. In both cases, minimum sensitivity is obtained at an azimuth separation angle of 90°. Target position sensitivity to sensor azimuth measurement error varies from 0.087:1 at a 90° separation angle to 0.35:1 at 20° and 160° separation angles. This means that at a 90° separation angle, a 1 degree azimuth measurement error will result in a 0.087 unit target position error.

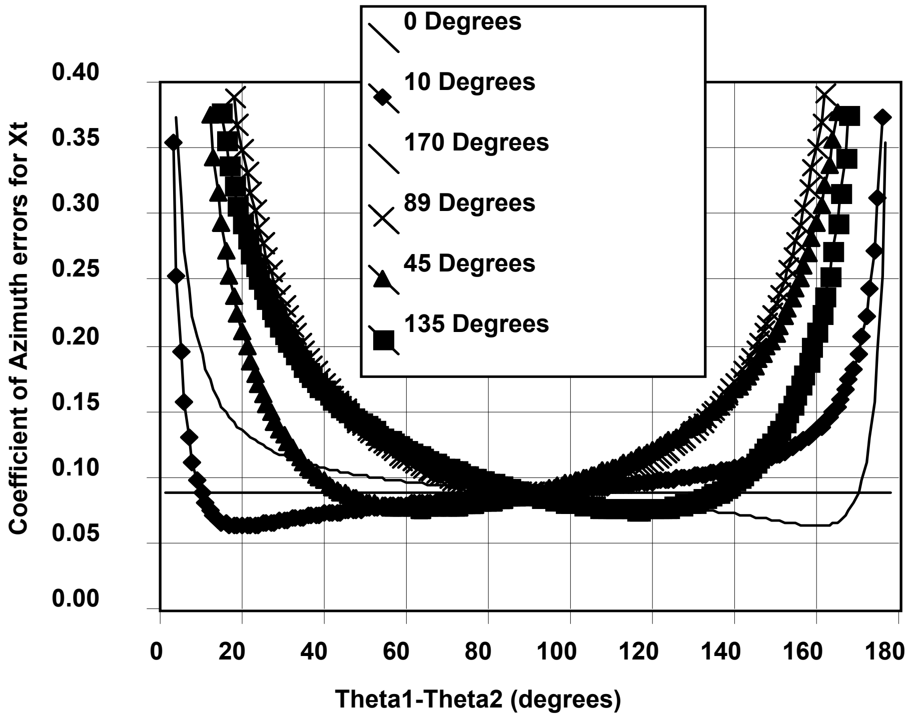

Figure 4. Coefficient of σ_θ in equation (22) for σ_x as a function of azimuth separation angle (θ_1 - θ_2) for various values of sensor 1 azimuth angle (θ_1). Sensor to target range is 5 units. Coefficient is for azimuth errors measured in degrees.

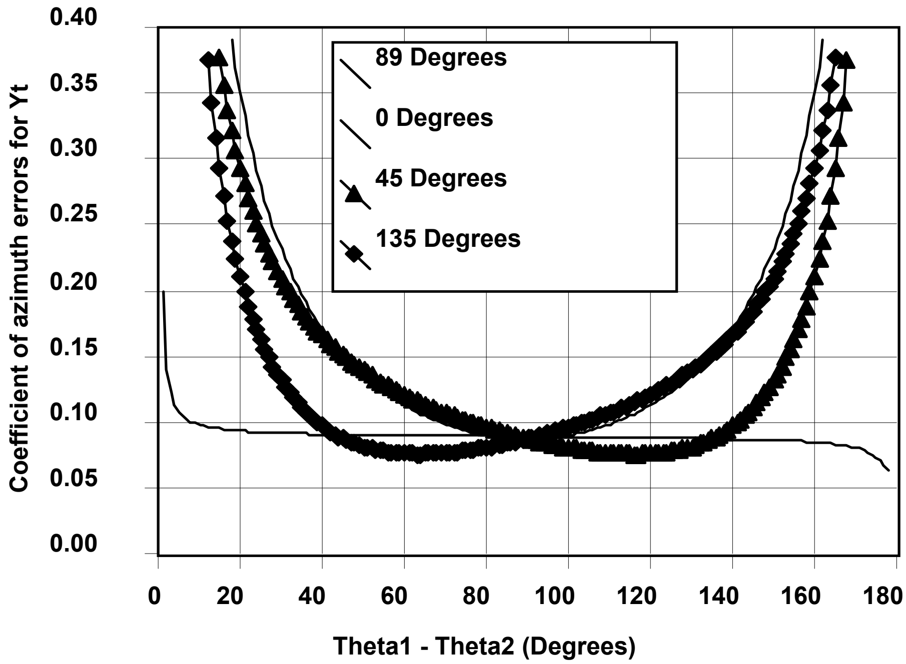

Figure 5. Coefficient of σ_θ in equation (23) for σ_y as a function of azimuth separation angle (θ_1 - θ_2) for various values of sensor 1 azimuth angle (θ_1). Sensor to target range is 5 units. Coefficient is for azimuth errors in degrees.

From equations (9) (11), (22) and (23), we can see that the coefficients for σ_{pos} do not depend on sensor location and hence have no range dependence. However, the coefficients for σ_θ do depend on the sensor locations and hence do have a range dependence. A plot (not shown to conserve space) of the coefficients of σ_θ in equations (22) and (23) shows an exact one-to-one scaling with range. Thus the coefficient of σ_θ in equation (22, 23), for a range of 2r, is equal to twice the coefficient at a range of r (coef:σ_θ(2r) = 2* coef:σ_θ(r)). The plots of the σ_θ coefficients (figures 4 and 5) are for a range of 5 units. Thus to scale to other ranges, one would divide by 5 and multiply by the appropriate range.

Errors in our estimation of the sensor to target range (σ_r, equation 24) affect the accuracy of our estimate of the target z location (σ_z, equation 20). Equation 24 describes the relationship between error measurements in the sensor locations and azimuth angles to the estimate of the sensor to target range. The coefficient of σ_{pos} (range error as a function of sensor location error) is independent of the target to sensor range and almost independent of absolute angle, exhibiting only a dependence on the angle between sensors (figure 6). The coefficient of σ_θ (range error as a function of sensor azimuth angle error) is almost independent of absolute angle, but has a direct linear scaling with range. Figure 7 shows a plot of the coefficient of σ_θ (for azimuth errors in degrees) as a function of azimuth separation angle for a range of 5 units.

From figure 6 we see a broad flat region of minimum sensitivity of 1.4:1 between separation angles of 60° and 120°. The sensitivity then increases to 4:1 at angles of 20°and 160°. From figure 7 we see the range sensitivity to azimuth measurement errors varies between 0.087:1 at a separation angle of 90°, to a sensitivity of 0.33 range units per degree of azimuth error at separation angles of 20°and 160° (at a range of 5 units).

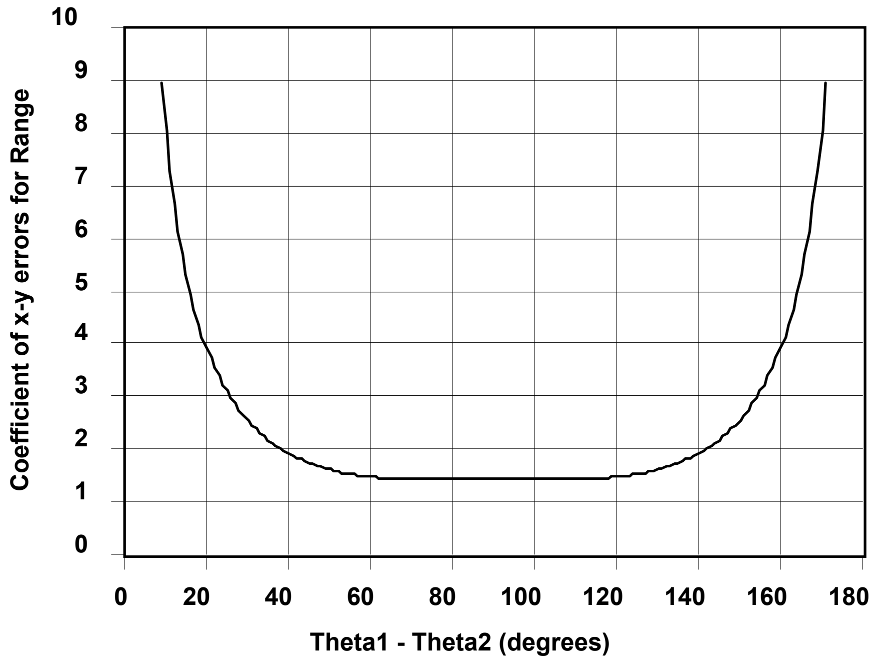

Figure 6. Coefficient of σ_{pos} in equation (24) for σ_r as a function of azimuth separation angle (θ_1 - θ_2).

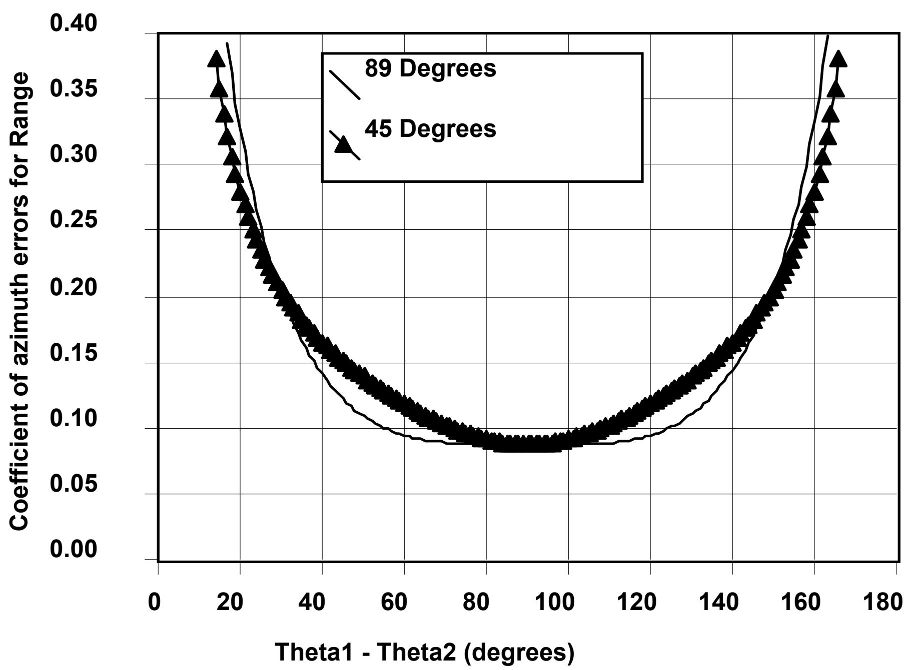

Figure 7. Coefficient of σ_θ in equation (24) for σ_r as a function of azimuth separation angle (θ_1 - θ_2). Sensor to target range is 5 units. Coefficient is for azimuth errors in degrees.

Next we examine the behavior of the coefficients of σ_r and σ_φ in equation (20). As given by equation (16), the coefficient of σ_φ is $\partial z_t/\partial\varphi_i = r_i\sec^2\varphi_i$. This obviously scales linearly with sensor to target range (r_i). A plot of this coefficient (in units per degree) as function of φ_i, for fixed range (5 units) is shown in figure 8. The coefficient of σ_r is simply $\partial z_t/\partial r_i = \tan\varphi_i$, as given by equation (16). Since this can be looked up in many standard references, or computed trivially, the graph is omitted in order to save space.

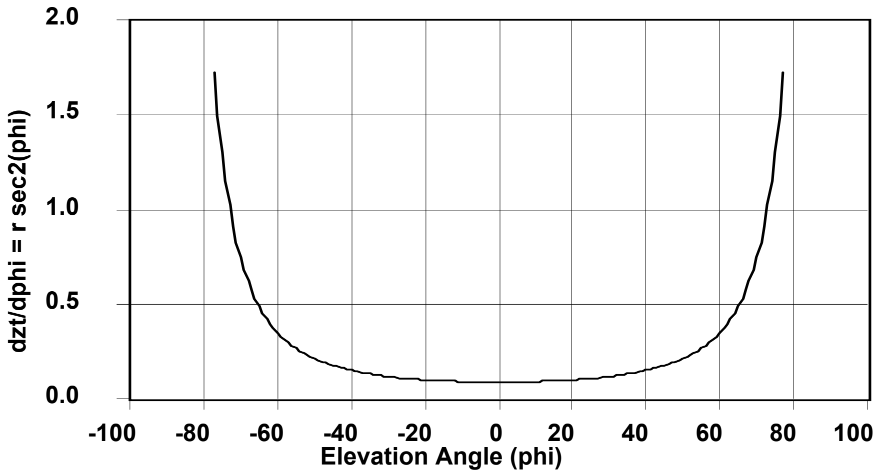

Figure 8. Coefficient of σ_φ in equation (20) as a function of elevation angle (φ) for a sensor to target range of 5 units. Coefficient is for elevation errors in degrees.

3.3 Monte-Carlo Results

The first set of 1000 runs was performed using a sensor position measurement error with a standard deviation of 1.0 unit. A separate set of random numbers was generated for both the x and y sensor location for each sensor. The second set of 1000 runs was performed using a sensor azimuth angle measurement error with a standard deviation of 1 degree. A separate set of random numbers was generated for each sensor. As already mentioned, all 4 sensors were at a range of 5 units from the target, but with azimuth angles of 40°, 50°, 130°, and 140°. Table 1 gives the standard deviation of the target x and y location and sensor 1 to target range for the first set of runs (sensor position measurement error). Table 2 gives the same information for the second set of runs (sensor azimuth angle measurement errors).

Table 1. Sensor position measurement error, standard deviation = 1.0 unit

Sensor	σ_x	σ_y	σ_r
1 & 2 (40-50°)	5.74	5.73	5.70
1 & 3 (40-130°)	1.00	0.96	1.41
1 & 4 (40-140°)	0.93	1.08	1.50
2 & 3 (50-130°)	1.07	0.90	1.44
2 & 4 (50-140°)	0.99	0.99	1.43
3 & 4 (130-140°)	5.68	5.66	5.65

Table 2. Sensor azimuth measurement error, standard deviation = 1.0 degree

Sensor	σ_x	σ_y	σ_r
1 & 2 (40-50°)	0.555	0.555	0.780
1 & 3 (40-130°)	0.087	0.088	0.088
1 & 4 (40-140°)	0.079	0.097	0.091
2 & 3 (50-130°)	0.095	0.081	0.090
2 & 4 (50-140°)	0.086	0.089	0.088
3 & 4 (130-140°)	0.554	0.551	0.776

We can compare the Monte Carlo results presented here with the predictions of the analytically computed graphs in the last section. If we compare line 2 (sensor pair 1 and 3, 90° separation, θ_1 = 40°) in both tables, with the graphs, we find excellent agreement. At a separation of 90°, figures 2 and 3 predict a standard deviation for both x and y of 1, compared to table 1 values of 1.00 and 0.96. Figures 3 and 4 predict a standard deviation for both x and y of about 0.087, compared to table 2 values of 0.087 and 0.088. Similar comparisons can be made for σ_r using figures 6 and 7. We can compare the line 1 (sensor pair 1 and 2, 10° separation, θ_1 = 40°) to the graphs using the θ_1 = 45° data set as an approximation. Again good agreement is found in all cases.

4. Discussion

In this section, we summarize what we have learned and then apply it to two cases: 1) a laboratory application and 2) a test range application.

The basic equations for estimating target position in three dimensions (x, y, z) are given in section 2.1. The target (x,y) position can be estimated using either equation (4) for an N sensor problem, or equations (6) and (7) when using two sensors at a time. From this result, one estimates the horizontal sensor to target range for each sensor, using equation (3). Finally, one estimates the target z location, using equation (5).

Section 2.2 develops the basic error propagation equations which allow one to determine the affect on the position estimate, resulting from specific errors in the sensor location and pointing measurements. Equations (8) - (11) give the error propagation equations and coefficients for estimates of the target (x,y) location. Equations (13), (14) give the error propagation equation and coefficients for estimates of the sensor to target range, and equations (15), (16) give the error propagation equations and coefficients for estimates of the target z location.

While section 2.2 allows one to determine the effect of a specific measurement error on the target position estimate, section 2.3 introduces gaussian measurement error and allows one to relate the standard deviation in sensor location and orientation measurements to a standard deviation in the target location estimate. In this model, there are three measurement uncertainties, the standard deviation of sensor (x,y) location measurements σ_{pos}, the standard deviation of sensor azimuth measurements σ_θ, and the standard deviation of sensor elevation angle measurements σ_φ. Given these uncertainties, and approximate sensor locations, azimuth angles, and elevation angles, one can estimate the standard deviation of the target position estimates using equations (22)-(24) and (20).

Section 3.2 presents graphs of the error propagation coefficients relating the standard deviation of measurement errors to the standard deviation of target position estimates (from equations 20-24). Examining figures (2)-(7) we see that estimates of the target (x,y) location and hence sensor to target range, have a strong dependence on the azimuth angle between sensors, with a minimum error propagation occurring at an angular separation of 90°. In general, the error propagation curves become steeper as the sensor to sensor azimuth separation decreases below 90° or increases above 90°, becoming infinite at 0° and 180°. As a general rule of thumb, the curves appear to be relatively flat between 60° and 120°, thus providing a preferred range of orientations. The curves become quite steep for separations less than 20° and greater than 160°, indicating that orientations in these ranges are to be avoided. Another interesting observation is the sensitivity to the absolute sensor orientation shown in figures 2-5. What this sensitivity indicates is that one can minimize the sensitivity of either the x or the y target location (to sensor location and orientation errors), at the expense of increased sensitivity in the other dimension.

While the coefficients of sensor location measurement errors (figures 2, 3, 6) depend primarily on sensor to sensor azimuth separation angle, and not on sensor to target range, the coefficients of sensor azimuth angle measurement errors (figures 4, 5, 7) depend linearly on the range. The figures shown here are for a range of 5 units. To apply these graphs, they must be scaled by the appropriate range. Thus divide by 5 and multiply by the sensor to target range in your system of units. Reading from the graphs, at a separation angle of 90°, there is a 1:1 relation between sensor location measurement errors and target position estimation, while there is a 0.087 unit error per degree of azimuth angle error (at a range of 5 units).

Finally, figure 8 shows the sensitivity of the target z position estimate to errors in the measurement of the sensor elevation angle. The sensitivity is relatively flat between elevation angles of -40° and +40° with a minimum at 0° degrees elevation. The sensitivity increases rapidly at elevation angles below -60° and above +60°.

Section 3.3 uses a Monte Carlo approach to numerically verify the error coefficients derived in Section 2.3 and graphed in Section 3.2. Very good agreement is obtained, thus giving confidence in the mathematical formulation and the software implementation, as expressed in the graphical results. This independent validation gives confidence that the graphical results of Section 3.2 are correct.

We now turn to the two example cases (laboratory and test range), in order to demonstrate how to utilize the results of this paper in a practical application. For a typical automotive crash test application, one might want to measure a 3D position to an accuracy of 5 mm, using 2 sensors at a range of 4 meters from the target area. One is typically constrained on the sensor separation, so we will assume a sensor azimuth separation of 40°. Referring to figure 2, if sensor 1 has an azimuth angle of 45° to the target, then for the 40° separation, our error propagation coefficient to the target x position, for sensor position errors is 1.11. Similarly, for the target y position, the coefficient is 1.90. The error coefficient for azimuth errors is 0.097*4/5 = 0.078 m/degree for the x measurement and 0.166*4/5 = 0.133 m/degree for the y measurement. The 4/5 ratio is to scale the range from the 5 unit range used in this paper, to the 4 meter range in the example. If we divide our error equally between location measurement and azimuth measurement (allocating 2.5 mm of error to each), then we need to measure the sensor location to 1.3 mm (2.5 mm / 1.90) accuracy in the laboratory frame of reference. We need to measure the azimuth pointing angle to 0.02 degrees = 0.35 mrad (0.0025 m / 0.133 m/degree). If we use a 512x512 pixel sensor with a 20° field of view, then the IFOV (pixel field of view) is 0.04°. This implies we need to measure target locations to better than ½ a pixel accuracy. The preceding sensor location and pointing accuracy requirements are those necessary to compute an absolute 3D position for the target. If on the other hand, one only needs to compute target position relative to other objects in the field of view, then accuracy of the absolute sensor location and pointing angle are not important (as long as the sensor is stationary). However, the pixel accuracy requirement still obtains.

For a range application, one may use a tracking mount, as opposed to a static mount with fixed pointing. A typical accuracy might be 2 meters at a range of 5 km. In this case, we might assume a 0.25° field of view with a 512x512 sensor, giving 1.76 arc sec per pixel or 8.5 μrad per pixel. For this example, we will assume an optimal 90° azimuth separation angle geometry. The coefficient of sensor location measurement errors (figures 2, 3) is 1.0, while the coefficient of azimuth angle measurement errors is 0.087 * (5/5) = 0.087 km per degree of error (figures 4, 5). Again dividing the errors equally between sensor location and azimuth angle errors, we see that must know the sensor position to an accuracy of 1 m (quite doable with differential GPS technology). We must know our azimuth pointing angles to an accuracy of 0.01° (1 m / 87 m/degree) = 17 μrad. The estimate of the target z position depends on the sensor z location measurement, the accuracy of the sensor elevation angle measurement, and the accuracy of the sensor to target range. If we assume a 40° elevation angle, then from figure 8, the elevation angle error coefficient is 0.2 km/degree. Thus we need to know the elevation angle to 0.005° to have a 1 m accuracy.

5. Conclusions

This paper has developed the error propagation equations for use with two, angles-only sensors to perform a 3D position estimation via triangulation. The analytical error propagation equations developed in section 2 and graphed in section 3.2 have been verified using a Monte-Carlo approach in section 3.3.

The optimum geometry for a two sensor system is to have a 90° azimuth separation angel between the two sensors. Target (x,y) position estimation accuracy depends on the accuracy of the sensor location measurement, the sensor azimuth pointing measurement, the sensor to target range, and the sensor-to-sensor azimuth separation geometry. Target z position estimation accuracy depends on sensor to target horizontal range (and hence all of the preceding components of the (x,y) position estimation), the sensor z location measurement accuracy, and the sensor elevation angle measurement.

The equations and graphs in this paper give practical tools for designing a two-sensor 3D position estimation system to achieve a desired level of accuracy. They are also useful for estimating the accuracy of an existing system. Future work should extent the current analysis to a full multi-sensor (more than two) system.

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